Simple Formulae for Designing Classical Recursive Digital Filters

Written by Tapio Saramäki based on his lecture notes
http://www.cs.tut.fi/~ts/part4.pdf and
S. Darlington, "Simple algorithms for elliptic filters and generalizations thereof", IEEE Trans. Circuits and Systems, CAS-25, pp. 975-980, Dec. 1978.

Part I: DESIGN OF Lowpass filters

I. Filter Transfer Function and Criteria Under Consideration

Consider a digital filter characterized by the following Nth-order transfer function:

$$H(z) = \kappa \prod_{\ell=1}^{N}(1-\alpha_\ell z^{-1}) \Big/ \prod_{\ell=1}^{N}(1-\beta_\ell z^{-1}) \tag{1}$$

and let the specifications be stated as

$$\begin{aligned} 0 \le \text{Loss}\left(H(e^{j\omega})\right) \le A_p \quad &\text{for } \omega \in [0, \omega_p] \\ A_s \le \text{Loss}\left(H(e^{j\omega})\right) \quad &\text{for } \omega \in [\omega_s, \pi], \end{aligned} \tag{2}$$

where

$$\text{Loss}\left(H(e^{j\omega})\right) = -10\log_{10}\left|H(e^{j\omega})\right|^2 \equiv -20\log_{10}\left|H(e^{j\omega})\right|. \tag{3}$$

II. Minimum Filter Orders

Initialize

$$a_0 = \sqrt{\tan(\omega_s/2)/\tan(\omega_p/2)} \text{ and } J_0 = \sqrt[4]{(10^{A_s/10}-1)/\,(10^{A_p/10}-1)}. \tag{4}$$

and evaluate recursively

$$a_1 = {a_0}^2 + \sqrt{{a_0}^4 - 1}, a_2 = {a_1}^2 + \sqrt{{a_1}^4 - 1}, a_3 = {a_2}^2 + \sqrt{{a_2}^4 - 1},$$
$$a_4 = {a_3}^2 + \sqrt{{a_3}^4 - 1} \tag{5}$$

and

$$J_1 = {J_0}^2 + \sqrt{{J_0}^4 - 1}, J_2 = {J_1}^2 + \sqrt{{J_1}^4 - 1}, J_3 = {J_2}^2 + \sqrt{{J_2}^4 - 1},$$
$$J_4 = {J_3}^2 + \sqrt{{J_3}^4 - 1}. \tag{6}$$

The minimum orders for Butterworth, both Chebyshev and inverse Chebyshev, and Cauer filters are then the smallest integers satisfying [1]

$$\begin{aligned} N_{\min}^{(\text{Butter})} &\geq \log_{10}(J_0)/\log_{10}(a_0) \\ N_{\min}^{(\text{Cheby})} \equiv N_{\min}^{(\text{invChe})} &\geq \log_{10}(J_1)/\log_{10}(a_1) \\ N_{\min}^{(\text{Cauer})} &\geq \log_{10}(2J_4)/\log_{10}(2a_4). \end{aligned} \tag{7}$$

III. Minimized Values of ω_s and Maximized Values of ω_p for the Minimum[2] Filter Orders

Initialize[3]

$$\hat{a}_4 = \left[J_4/2^{\left(N_{\min}^{(\text{Cauer})}-1\right)}\right]^{1/N_{\min}^{(\text{Cauer})}} \tag{8}$$

and evaluate recursively

$$\hat{a}_3 = \sqrt{(\hat{a}_4 + 1/\hat{a}_4)/2}, \hat{a}_2 = \sqrt{(\hat{a}_3 + 1/\hat{a}_3)/2}, \hat{a}_2 = \sqrt{(\hat{a}_3 + 1/\hat{a}_3)/2},$$
$$\hat{a}_0 = \sqrt{(\hat{a}_1 + 1/\hat{a}_1)/2}. \tag{9}$$

The minimized stopband edge angles are then

$$\begin{aligned} \widehat{\omega}_s^{(\text{Butter})} &= 2\tan^{-1}\left[J_0^{\frac{2}{N_{\min}^{(\text{Butter})}}}\tan\left(\frac{\omega_p}{2}\right)\right] \\ \widehat{\omega}_s^{(\text{Cheby})} \equiv \widehat{\omega}_s^{(\text{invChe})} &= 2\tan^{-1}\left[\frac{1}{2}\left(J_1^{\frac{1}{N_{\min}^{(\text{Cheby})}}} + 1/J_1^{\frac{1}{N_{\min}^{(\text{Cheby})}}}\right)\tan\left(\frac{\omega_p}{2}\right)\right] \\ \widehat{\omega}_s^{(\text{Cauer})} &= 2\tan^{-1}\left[{\hat{a}_0}^2\tan\left(\frac{\omega_p}{2}\right)\right]. \end{aligned} \tag{10}$$

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[1] The filter meets exactly the criteria with a "decimal" order. Hence, the integer larger than or equal to this decimal order has to be used. This gives additional margin for the filter design. As will be seen next, this additional margin can be allocated to either minimizing the values of ω_s or A_p or maximizing the values of ω_p or A_s.

[2] The following formulae apply, in addition to the minimum filter orders, to any orders.

[3] In (4), (5), and (6), a_0 to a_4 have been evaluated. Now in (8) and (9), a_4 to a_0 are calculated. This explains the use of hat even though the same quantities are under consideration.

and corresponding maximized passband edge angles are

$$\hat{\omega}_p^{(\mathrm{Butter})} = 2\tan^{-1}\left[\tan\left(\frac{\omega_s}{2}\right)/J_0^{\frac{2}{N_{\min}^{(\mathrm{Butter})}}}\right]$$

$$\hat{\omega}_p^{(\mathrm{Cheby})} \equiv \hat{\omega}_s^{(\mathrm{invChe})} = 2\tan^{-1}\left\{\tan\left(\frac{\omega_s}{2}\right)\Big/\left[\frac{1}{2}\left(J_1^{\frac{1}{N_{\min}^{(\mathrm{Cheby})}}} + 1/J_1^{\frac{1}{N_{\min}^{(\mathrm{Cheby})}}}\right)\right]\right\} \tag{11}$$

$$\hat{\omega}_p^{(\mathrm{Cauer})} = 2\tan^{-1}\left[\tan\left(\frac{\omega_s}{2}\right)/\hat{a}_0{}^2\right].$$

IV. Maximized Values of A_s and Minimized Values of A_p for the Minimum Filter Orders

Initialize[4]

$$\hat{J}_4 = 2^{\left(N_{\min}^{(\mathrm{Cauer})}-1\right)} a_4{}^{N_{\min}^{(\mathrm{Cauer})}} \tag{12}$$

and evaluate recursively

$$\hat{J}_3 = \sqrt{(\hat{J}_4 + 1/\hat{J}_4)/2},\ \hat{J}_2 = \sqrt{(\hat{J}_3 + 1/\hat{J}_3)/2},\ \hat{J}_1 = \sqrt{(\hat{J}_2 + 1/\hat{J}_2)/2},$$
$$\hat{J}_0 = \sqrt{(\hat{J}_1 + 1/\hat{J}_1)/2}. \tag{13}$$

The maximized stopband attenuations are then

$$\hat{A}_s^{(\mathrm{Butter})} = 10\log_{10}\left[1 + \left(10^{A_p/10} - 1\right) a_0{}^{4N_{\min}^{(\mathrm{Butter})}}\right]$$

$$\hat{A}_s^{(\mathrm{Cheby})} \equiv \hat{A}_s^{(\mathrm{invChe})}$$

$$= 10\log_{10}\left[1 + \left(10^{A_p/10} - 1\right)\left[\frac{1}{2}\left(a_1{}^{N_{\min}^{(\mathrm{Cheby})}} + 1/a_1{}^{N_{\min}^{(\mathrm{Cheby})}}\right)\right]^2\right] \tag{14}$$

$$\hat{A}_s^{(\mathrm{Cauer})} = 10\log_{10}\left[1 + \left(10^{A_p/10} - 1\right)\hat{J}_0{}^4\right]$$

and the corresponding minimized passband variations are

$$\hat{A}_p^{(\mathrm{Butter})} = 10\log_{10}\left[1 + \left(10^{A_s/10} - 1\right)/a_0{}^{4N_{\min}^{(\mathrm{But})}}\right]$$

$$\hat{A}_p^{(\mathrm{Cheby})} \equiv \hat{A}_p^{(\mathrm{invChe})} = 10\log_{10}\left[1 + \frac{\left(10^{A_s/10} - 1\right)}{\left[\frac{1}{2}\left(a_1{}^{N_{\min}^{(\mathrm{Cheby})}} + 1/a_1{}^{N_{\min}^{(\mathrm{Cheby})}}\right)\right]^2}\right] \tag{15}$$

$$\hat{A}_p^{(\mathrm{Cauer})} = 10\log_{10}\left[1 + \left(10^{A_s/10} - 1\right)/\hat{J}_0{}^4\right].$$

[4]. The use of hat is because J_4 to J_0 be evaluated in comparison with equations (4) and (6), where J_0 to J_4 are calculated.

V. Filter Poles, Zeros, and Scaling Constant

After knowing

- $\omega_p^{(\mathrm{Butter})}$, $A_p^{(\mathrm{Butter})}$, and $N^{(\mathrm{Butter})}$;
- $\omega_p^{(\mathrm{Cheby})}$, $A_p^{(\mathrm{Cheby})}$, and $N^{(\mathrm{Cheby})}$;
- $\omega_p^{(\mathrm{invChe})}$, $\omega_s^{(\mathrm{invCheby})}$, $A_p^{(\mathrm{invChe})}$, and $N^{(\mathrm{invCheby})}$; and
- $\omega_p^{(\mathrm{Cauer})}$, $\omega_s^{(\mathrm{Cauer})}$, $A_p^{(\mathrm{Cauer})}$, and $N^{(\mathrm{Cauer})}$,

the poles and zeros as well as the scaling constant can be found as follows for each filter type. The use of the above input data will be indicated by **bold red**.

Butterworth Filter

The filter zeros and poles are

$$\alpha_\ell = -1 \text{ for } \ell = 1, 2, \dots, N^{(\mathrm{Butter})} \qquad (16a)$$
$$\beta_\ell = (1 + p_\ell)\,/(1 - p_\ell) \text{ for } \ell = 1, 2, \dots, N^{(\mathrm{Butter})},$$

where

$$p_\ell = \tan\left(\frac{\boldsymbol{\omega_p^{(\mathrm{Butter})}}}{2}\right)\left(\frac{1}{10^{\boldsymbol{A_p^{(\mathrm{Butter})}}/10} - 1}\right)^{\frac{1}{2\boldsymbol{N^{(\mathrm{Butter})}}}} \cdot e^{j\pi\left(\frac{1}{2}+\frac{2\ell-1}{2\boldsymbol{N^{(\mathrm{Butter})}}}\right)} \qquad (16b)$$

and the scaling constant is

$$\kappa = \prod_{\ell=1}^{N}(1 - \alpha_\ell z^{-1}) \Big/ \prod_{\ell=1}^{N}(1 - \beta_\ell z^{-1}). \qquad (16c)$$

Chebyshev Filter

The filter zeros and poles are

$$\alpha_\ell = -1 \text{ for } \ell = 1, 2, \dots, N^{(\mathrm{Cheby})} \qquad (17a)$$
$$\beta_\ell = (1 + p_\ell)\,/(1 - p_\ell) \text{ for } \ell = 1, 2, \dots, N^{(\mathrm{Cheby})},$$

where

$$p_\ell = \tan\left(\frac{\boldsymbol{\omega_p^{(\mathrm{Cheby})}}}{2}\right)\left(\frac{\hat{p}_\ell - 1/\hat{p}_\ell}{2}\right) \qquad (17b)$$

with

$$\hat{p}_\ell = \left(\sqrt{\frac{1}{10^{A_p^{(\text{Cheby})}/10}-1}} + \sqrt{\frac{1}{10^{A_p^{(\text{Cheby})}/10}-1}+1}\right)^{\frac{1}{N^{(\text{Cheby})}}} \cdot e^{j\pi\left(\frac{1}{2}+\frac{2\ell-1}{2N^{(\text{Cheby})}}\right)}. \tag{17c}$$

Finally, the scaling constant is

$$\kappa = \varrho \frac{\prod_{\ell=1}^{N}(1-\alpha_\ell z^{-1})}{\prod_{\ell=1}^{N}(1-\beta_\ell z^{-1})}, \tag{17d}$$

where

$$\varrho = \begin{cases} 10^{-A_p^{(\text{Cheby})}/20} & \text{for } N^{(\text{Cheby})} \text{ even} \\ 1 & \text{for } N^{(\text{Cheby})} \text{ odd.} \end{cases} \tag{17e}$$

Inverse Chebyshev Filter

The filter poles are[5]

$$\beta_\ell = -(1+p_\ell)\,/\,(1-p_\ell) \ \text{ for } \ \ell = 1, 2, \ldots, N^{(\text{invCheby})}, \tag{18a}$$

where

$$p_\ell = \tan\left(\frac{\pi - \omega_s^{(\text{invChe})}}{2}\right)\left(\frac{\hat{p}_\ell - 1/\hat{p}_\ell}{2}\right) \tag{18b}$$

with

$$\hat{p}_\ell = \left(\sqrt{\frac{1}{10^{\hat{A}_p/10}-1}} + \sqrt{\frac{1}{10^{\hat{A}_p/10}-1}+1}\right)^{1/N^{(\text{invChe})}} \cdot e^{j\pi\left(\frac{1}{2}+\frac{2\ell-1}{2N^{(\text{invChe})}}\right)}. \tag{18c}$$

Here,

$$\hat{A}_p = -10\log_{10}\left(1 - 10^{\hat{A}_s/10}\right), \tag{18d}$$

where

$$\hat{A}_s = 10\log_{10}\left[1 + \left(10^{A_p^{(\text{invCheby})}/10} - 1\right)\left[\frac{1}{2}\left(\tilde{a}_1^{\,N^{(\text{invChe})}} + 1/\tilde{a}_1^{\,N^{(\text{invChe})}}\right)\right]^2\right] \tag{18e}$$

with

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[5] In fact, the β_ℓ's are determined as the poles of the power-complementary high-pass Chebyshev filter, which shares these poles with the desired low-pass inverse Chebyshev. The passband ripple for this high-pass filter is $\hat{A}_p$ in (18d). In the above, the $-\beta_\ell$'s are the poles of the low-pass Chebyshev filter with passband edge at $\pi - \omega_s^{(\text{invChe})}$, see (18b), and passband ripple of $\hat{A}_p$.This is also seen by comparing (18b) and (18c) with (17b) and (17c). These poles are then converted to the desired ones by using $z^{-1} \to -z^{-1}$, the simplest low-pass-to-high-pass transformation. For more detail about this transformation, see Part II of this contribution.

$$\tilde{a}_1 = \frac{\tan\left(\omega_s^{(\mathrm{invChe})}/2\right)}{\tan\left(\omega_p^{(\mathrm{invChe})}/2\right)} + \sqrt{\left[\frac{\tan\left(\omega_s^{(\mathrm{invChe})}/2\right)}{\tan\left(\omega_p^{(\mathrm{invChe})}/2\right)}\right]^2 - 1}. \tag{18f}$$

The zeros, in turn, are

$$\beta_\ell = \begin{cases} \exp\left[j2\tan^{-1}\left\{\dfrac{\tan\left(\omega_s^{(\mathrm{invChe})}/2\right)}{\cos\left[\dfrac{(2\ell-1)\pi}{2N^{(\mathrm{invChe})}}\right]}\right\}\right] & \text{for } \ell = 1,2,\ldots,\left\lfloor N^{(\mathrm{invChe})}/2\right\rfloor \\ -1 & \text{for } \ell = \left\lfloor N^{(\mathrm{invChe})}/2\right\rfloor + 1 \text{ and } N^{(\mathrm{invChe})} \text{ odd} \\ \exp\left[-j2\tan^{-1}\left\{\dfrac{\tan\left(\omega_s^{(\mathrm{invChe})}/2\right)}{\cos\left[\dfrac{(2(N^{(\mathrm{invChe})}-\ell)-1)\pi}{2N^{(\mathrm{invChe})}}\right]}\right\}\right] & \text{for } \ell = N^{(\mathrm{invChe})} - \left\lfloor\dfrac{N^{(\mathrm{invChe})}}{2}\right\rfloor, \ldots, N^{(\mathrm{invChe})}. \end{cases} \tag{19}$$

The scaling constant is given by (16c).

Cauer Filter

Following equations (4) and (5) as well as (8) and (9), evaluate first

$$a_0 = \sqrt{\tan(\omega_s^{(\mathrm{Cauer})}/2)/\tan(\omega_p^{(\mathrm{Cauer})}/2)},\ a_1 = a_0^{\ 2} + \sqrt{a_0^{\ 4} - 1},$$
$$a_2 = a_1^{\ 2} + \sqrt{a_1^{\ 4} - 1},\ a_3 = a_2^{\ 2} + \sqrt{a_2^{\ 4} - 1},\ a_4 = (a_3)^2 + \sqrt{(a_3)^4 - 1} \tag{20}$$

and

$$\hat{J}_4 = 2^{(N^{(\mathrm{Cauer})}-1)}(a_4)^{N^{(\mathrm{Cauer})}},\ \hat{J}_3 = \sqrt{(\hat{J}_4 + 1/\hat{J}_4)/2},\ \hat{J}_2 = \sqrt{(\hat{J}_3 + 1/\hat{J}_3)/2}$$
$$\hat{J}_1 = \sqrt{(\hat{J}_2 + 1/\hat{J}_2)/2}. \tag{21}$$

After that, calculate

$$S_{10} = \sqrt{\frac{1}{10^{A_p^{(\mathrm{Cauer})}/10} - 1}} + \sqrt{\frac{1}{10^{A_p^{(\mathrm{Cauer})}} - 1} + 1},\ S_{20} = \hat{J}_1 S_{10} + \sqrt{\left(\hat{J}_1 S_{10}\right)^2 - 1},$$
$$S_{30} = \hat{J}_2 S_{20} + \sqrt{\left(\hat{J}_2 S_{20}\right)^2 - 1},\ S_{40} = \hat{J}_3 S_{30} + \sqrt{\left(\hat{J}_3 S_{30}\right)^2 - 1} \tag{22}$$

and[6] for $\ell = 1,2,\ldots,N^{(\mathrm{Cauer})}$

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[6] It has turned out that selecting the angles of the $\hat{p}_l^{(5)}$'s as shown (23) gives the right poles of the Cauer filter. This issue has not been considered in detail in the article by Darlington.

$$
\begin{aligned}
p_l^{(5)} &= \left(\frac{\hat{J}_4}{S_{40}} + \sqrt{\left(\frac{\hat{J}_4}{S_{40}}\right)^2 + 1}\right)^{1/\boldsymbol{N}^{(\text{Cauer})}} \cdot e^{j\pi\left(\frac{1}{2}+\frac{2\ell-1}{2\boldsymbol{N}^{(\text{Cauer})}}\right)}, \\
p_l^{(4)} &= \frac{1}{2a_4}\left(p_l^{(5)} - 1/p_l^{(5)}\right), p_l^{(3)} = \frac{1}{2a_3}\left(p_l^{(4)} - 1/p_l^{(4)}\right), \\
p_l^{(2)} &= \frac{1}{2a_2}\left(p_l^{(3)} - 1/p_l^{(3)}\right), p_l^{(1)} = \frac{1}{2a_1}\left(p_l^{(2)} - 1/p_l^{(2)}\right) \\
p_l^{(0)} &= \tan\left(\frac{\boldsymbol{\omega}_{\boldsymbol{p}}^{(\text{Cauer})}}{2}\right)\left(\frac{p_l^{(1)} - 1/p_l^{(1)}}{2}\right).
\end{aligned}
\tag{23}
$$

The filter poles are then

$$
\beta_\ell = \left(1 + p_l^{(0)}\right)/\left(1 - p_l^{(0)}\right) \text{ for } \ell = 1, 2, \ldots, N^{(\text{Cauer})}. \tag{24}
$$

Evaluate for $\ell = 1, 2, \ldots, \lfloor N^{(\text{Cauer})}/2 \rfloor$

$$
\begin{aligned}
z_l^{(4)} &= \frac{a_4}{\cos\left[\frac{(2\ell-1)\pi}{2\boldsymbol{N}^{(\text{Cauer})}}\right]}, z_l^{(3)} = \frac{1}{2a_3}\left(z_l^{(4)} + 1/z_l^{(4)}\right), z_l^{(2)} = \frac{1}{2a_3}\left(z_l^{(3)} + 1/z_l^{(3)}\right), \\
z_l^{(1)} &= \frac{1}{2a_2}\left(z_l^{(2)} + 1/z_l^{(2)}\right), z_l^{(0)} = \tan\left(\frac{\boldsymbol{\omega}_{\boldsymbol{p}}^{(\text{Cauer})}}{2}\right)\left(\frac{z_l^{(1)} + 1/z_l^{(1)}}{2}\right).
\end{aligned}
\tag{25}
$$

The filter zeros are then

$$
\beta_\ell = \begin{cases}
\exp\left[j2\tan^{-1}\left(z_l^{(0)}\right)\right] & \text{for } \ell = 1, 2, \ldots, \lfloor N^{(\text{Cauer})}/2 \rfloor \\
-1 & \text{for } \ell = \lfloor N^{(\text{Cauer})}/2 \rfloor + 1 \text{ and } N^{(\text{Cauer})} \text{ odd} \\
\exp\left[-j2\tan^{-1}\left(z_{N^{(\text{Cauer})}+1-\ell}^{(0)}\right)\right] & \text{for } \ell = N^{(\text{Cauer})} - \left\lfloor \frac{N^{(\text{Cauer})}}{2} \right\rfloor, \ldots, N^{(\text{Cauer})}.
\end{cases}
\tag{26}
$$

The scaling constant is obtained from (17d) and (17e) after replacing $A_p^{(\text{Cheby})}$ by $\boldsymbol{A}_{\boldsymbol{p}}^{(\text{Cauer})}$.

Part II: DESIGN OF HighPass, BandPass, and BandStop filters

I. Preliminaries: Common Practice

It is a common practice to start the synthesis high-pass, band-pass, and band-stop filters by determining a prototype low-pass filter with transfer function

$$H_{\mathrm{LP}}(Z) = \mathrm{K}\prod_{\ell=1}^{M}(1-\mathrm{A}_\ell Z^{-1})\bigg/\prod_{\ell=1}^{M}(1-\mathrm{B}_\ell Z^{-1}) \tag{27}$$

to meet the specifications

$$\begin{aligned} 0 \le \mathrm{Loss}\left(H_{\mathrm{LP}}\left(e^{j\theta}\right)\right) \le A_p \ \text{ for } \theta \in \left[0, \theta_p\right] \\ A_s \le \mathrm{Loss}\left(H_{\mathrm{LP}}\left(e^{j\theta}\right)\right) \ \text{ for } \theta \in [\theta_s, \pi]. \end{aligned} \tag{28}$$

The second step is then to apply the transformation of the form

$$Z^{-1} = G(z^{-1}) \tag{29}$$

to $H_{\mathrm{LP}}(Z)$ to give the desired-transfer function in the forms

$$H(z) = H_{\mathrm{LP}}(Z)_{\,|Z^{-1}=G(z^{-1})} = \kappa\prod_{\ell=1}^{N}(1-\alpha_\ell z^{-1})\bigg/\prod_{\ell=1}^{N}(1-\beta_\ell z^{-1}). \tag{30}$$

The key idea is to select the transformation such the resulting filter meets

$$\begin{aligned} 0 \le \mathrm{Loss}\left(H\left(e^{j\omega}\right)\right) \le A_p \ \text{ for } \omega \in \Omega_p \\ A_s \le \mathrm{Loss}\left(H\left(e^{j\omega}\right)\right) \ \text{ for } \omega \in \Omega_s. \end{aligned} \tag{31}$$

Table I gives the passband and stopband regions Ω_p and Ω_s for the high-pass, band-pass, and band-stop filters and the relationships between the band edges. Table II, in turn, summarizes the transformations that exactly convert the passband region $\left[0, \theta_p\right]$ of the prototype filter into the desired passband region Ω_p in the high-pass, band-pass, and band-stop cases[7].

[7] For more detail, see, e.g., A. G. Constantinides, “Spectral transformations for digital filters,” *Proc. Inst. Elec. Eng.*, vol. 117, pp. 1585-1590, Aug. 1970.

Table I

Passband and stopband regions Ω_p and Ω_s for the high-pass, band-pass, and band-stop filters

Filter Type	Passband region Ω_p	Stopband region Ω_p	Relation beween the band eges
High-pass	$[\omega_p, \pi]$	$[0, \omega_s]$	$\omega_s < \omega_p$
Band-pass	$[\omega_{p1}, \omega_{p2}]$	$[0, \omega_{s1}] \cup [\omega_{s2}, \pi]$	$\omega_{s1} < \omega_{p1} < \omega_{p2} < \omega_{s2}$
Band-stop	$[0, \omega_{p1}] \cup [\omega_{p2}, \pi]$	$[\omega_{s1}, \omega_{s2}]$	$\omega_{p1} < \omega_{s1} < \omega_{s2} < \omega_{p2}$

Table II

Transformations converting the passband $[0, \theta_p]$ of the prototype $H_{LP}(Z)$ into the desired passband region Ω_p of $H(z)$

Filter Type	Transformation	Associated Parameters
High-pass	$Z^{-1} = -\dfrac{z^{-1} + \rho}{1 + \rho z^{-1}}$	$\rho = -\cos\left(\dfrac{\theta_p + \omega_p}{2}\right) \Big/ \cos\left(\dfrac{\theta_p - \omega_p}{2}\right)$
Band-pass	$Z^{-1} = -\dfrac{z^{-2} - \dfrac{2\rho K}{K+1} z^{-1} + \dfrac{K-1}{K+1}}{1 - \dfrac{2\rho K}{K+1} z^{-1} + \dfrac{K-1}{K+1} z^{-2}}$	$\rho = \cos\left(\dfrac{\omega_{p2} + \omega_{p1}}{2}\right) \Big/ \cos\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)$ $K = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\theta_p$
Band-stop	$Z^{-1} = \dfrac{z^{-2} - \dfrac{2\rho}{1+K} z^{-1} + \dfrac{1-K}{1+K}}{1 - \dfrac{2\rho}{1+K} z^{-1} + \dfrac{1-K}{1+K} z^{-2}}$	$\rho = \cos\left(\dfrac{\omega_{p2} + \omega_{p1}}{2}\right) \Big/ \cos\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)$ $K = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\theta_p$

II. Proposed Simplifications

In the prototype filter criteria of (28), θ_p and θ_s have not been prespecified. In the band-pass and band-stop cases, the most natural selection of θ_p for many reasons is

$$\theta_p = \pi/2. \tag{32}$$

This selection results in $\tan\theta_p \equiv 1$ in the in the parameter K in Table II. Further study results in the lowpass-to-bandpass and the lowpass-to-bandstop transformations of Table III, where the transformation parameters γ_1 and γ_2 depend exactly similarly on the passband edges ω_{p1} and ω_{p2}.

In addition to simplifying the overall design process, the above selection of θ_p combines the designs of bandpass and bandstop filters. Substituting $Z = e^{j\theta}$ and $z = e^{j\omega}$ in the lowpass-to-bandpass and lowpass-to-bandstop transformations of Table III leads, after some reasoning[8], to the following relations (the superscripts BP and BS refer to the bandpass and bandstop cases):

$$\theta = f_{BP}(\omega) = F(\omega) - \pi \tag{33a}$$

and

$$\theta = f_{BS}(\omega) = F(\omega) \tag{33b}$$

between θ, the angular frequency of the prototype filter, and ω, the angular frequency of the resulting filter. Here,

$$F(\omega) = 2\omega + 2\mathrm{arctan2}\big(y(\omega), x(\omega)\big), \tag{34a}$$

where

$$y(\omega) = \gamma_1 \sin\omega - \gamma_2 \sin 2\omega \text{ and } x(\omega) = 1 - \gamma_1 \cos\omega + \gamma_2 \cos 2\omega. \tag{34b}$$

Table III

Proposed simplified transformations. In the band-pass and band-stop cases, $\theta_p = \pi/2$.

Filter Type	Transformation	Associated Parameters
High-pass	$Z^{-1} = -z^{-1}$	
Band-pass	$Z^{-1} = -\dfrac{z^{-2} - \gamma_1 z^{-1} + \gamma_2}{1 - \gamma_1 z^{-1} + \gamma_2 z^{-2}}$	$\gamma_1 = \dfrac{2\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) + \sin\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$
Band-stop	$Z^{-1} = \dfrac{z^{-2} - \gamma_1 z^{-1} + \gamma_2}{1 - \gamma_1 z^{-1} + \gamma_2 z^{-2}}$	$\gamma_2 = \dfrac{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) - \sin\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) + \sin\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$

Figure 1 exemplifies the use of the above relation in the bandpass case with the following passband and stopband regions:

$$\Omega_p = [0.3\pi, 0.5\pi] \text{ and } \Omega_s = [0, 0.2\pi] \cup [0.6\pi, \pi].$$

It is seen from this figure that, after determining the parameters of the transformation according to Table III for $\theta_p = \pi/2$, the region $[-\pi, \pi]$ of $\left|H_{\mathrm{LP}}\left(e^{j\theta}\right)\right|$ is converted into the region$[0, \pi]$ of $\left|H\left(e^{j\omega}\right)\right|$ such that $\theta_p = -\pi/2$ and $\theta_p = \pi/2$ are, respectively, mapped to $\omega_{p1} = 0.3\pi$

[8] $F(\omega)$ in (34a) and (34b) is the phase response of the second-order all-pass filter with transfer function $(z^{-2} - \gamma_1 z^{-1} + \gamma_2)/(1 - \gamma_1 z^{-1} + \gamma_2 z^{-2})$. In the bandstop case, this $F(\omega)$ is directly used in (33b). In (33a), $-\pi$ has been included due to the minus sign in the corresponding transformation.

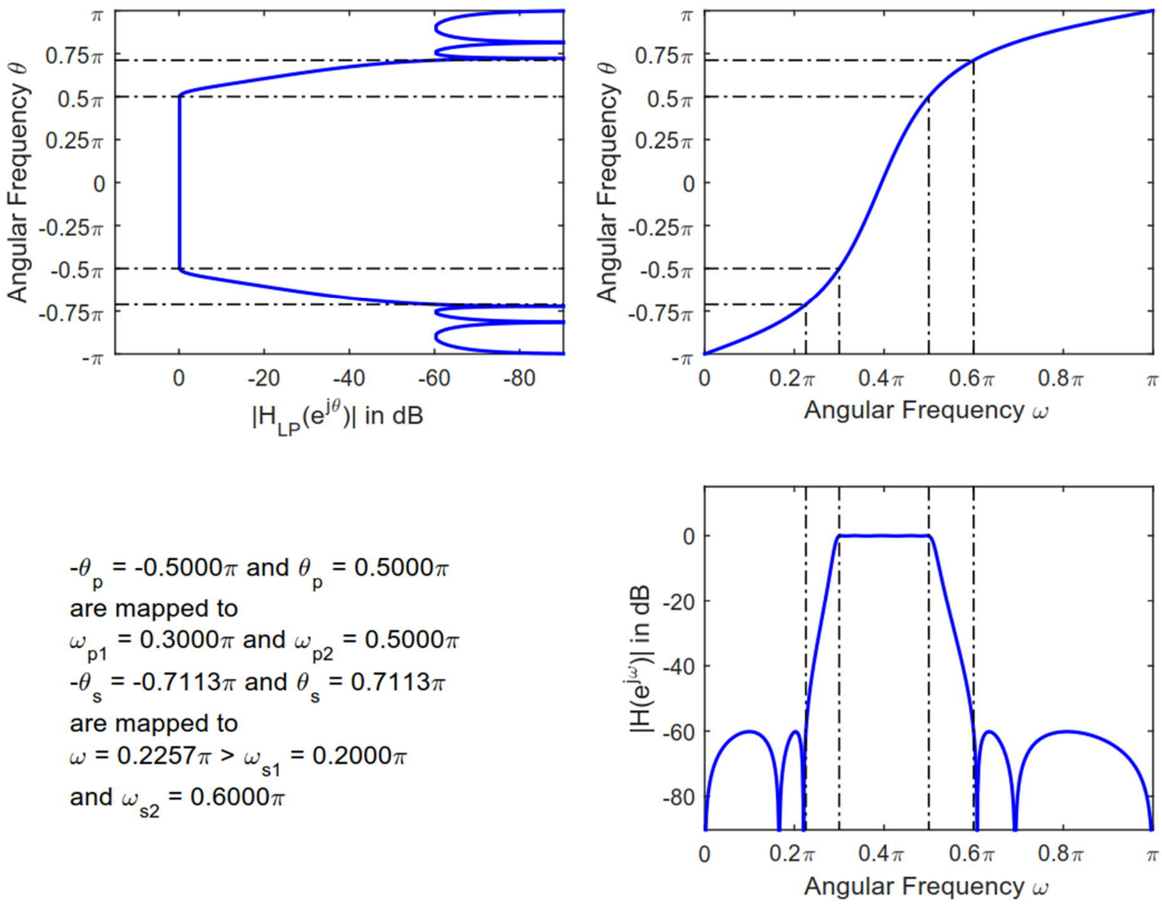

Figure 1. Design of a band-pass filter with $\Omega_p = [0.3\pi, 0.5\pi]$ and $\Omega_s = [0, 0.2\pi] \cup [0.6\pi, \pi]$ using the lowpass-to-bandpass transformation for $\theta_p = \pi/2$. $\theta_s = 0.711253\pi$ is required.

and $\omega_{p2} = 0.5\pi$. Hence, the desired passband property $0 \leq \text{Loss}\left(H_{\text{LP}}\left(e^{j\theta}\right)\right) \leq A_p$ on $\left[-\theta_p, \theta_p\right]$ is transferred to $0 \leq \text{Loss}\left(H\left(e^{j\omega}\right)\right) \leq A_p$ on $\left[\omega_{p1}, \omega_{p2}\right]$.

In order to convert $A_s \leq \text{Loss}\left(H_{\text{LP}}\left(e^{j\theta}\right)\right)$ on $[-\pi, -\theta_s]$ and $[\theta_s, \pi]$ to $A_s \leq \text{Loss}\left(H\left(e^{j\omega}\right)\right)$ to $A_s \leq \text{Loss}\left(H\left(e^{j\omega}\right)\right)$ on $[0, \omega_{s1}]$ and $[\omega_{s2}, \pi]$, $-\theta_s$ should be mapped to $\omega_{s1} = 0.2\pi$ or to an angular frequency larger than ω_{s1} and θ_s to $\omega_{s2} = 0.6\pi$ or to an angular frequency smaller than ω_{s2}. In terms of the function $f_{BP}(\omega)$ in (33a), the requirement for θ_s becomes

$$\theta_s = \min\{|f(\omega_{s1})|, f(\omega_{s2})\}. \tag{35}$$

In Fig. 1, $\theta_s = 0.711253\pi$ is mapped to $\omega_{s2} = 0.6\pi$, whereas $-\theta_s$ is mapped to $\omega = 0.225713\pi > \omega_{s1} = 0.2\pi$ so that the first stopband region becomes wider than needed.

Figure 2, in turn, concentrates on the bandstop case with

$$\Omega_p = [0.3\pi, 0.5\pi] \text{ and } \Omega_s = [0, 0.2\pi] \cup [0.6\pi, \pi].$$

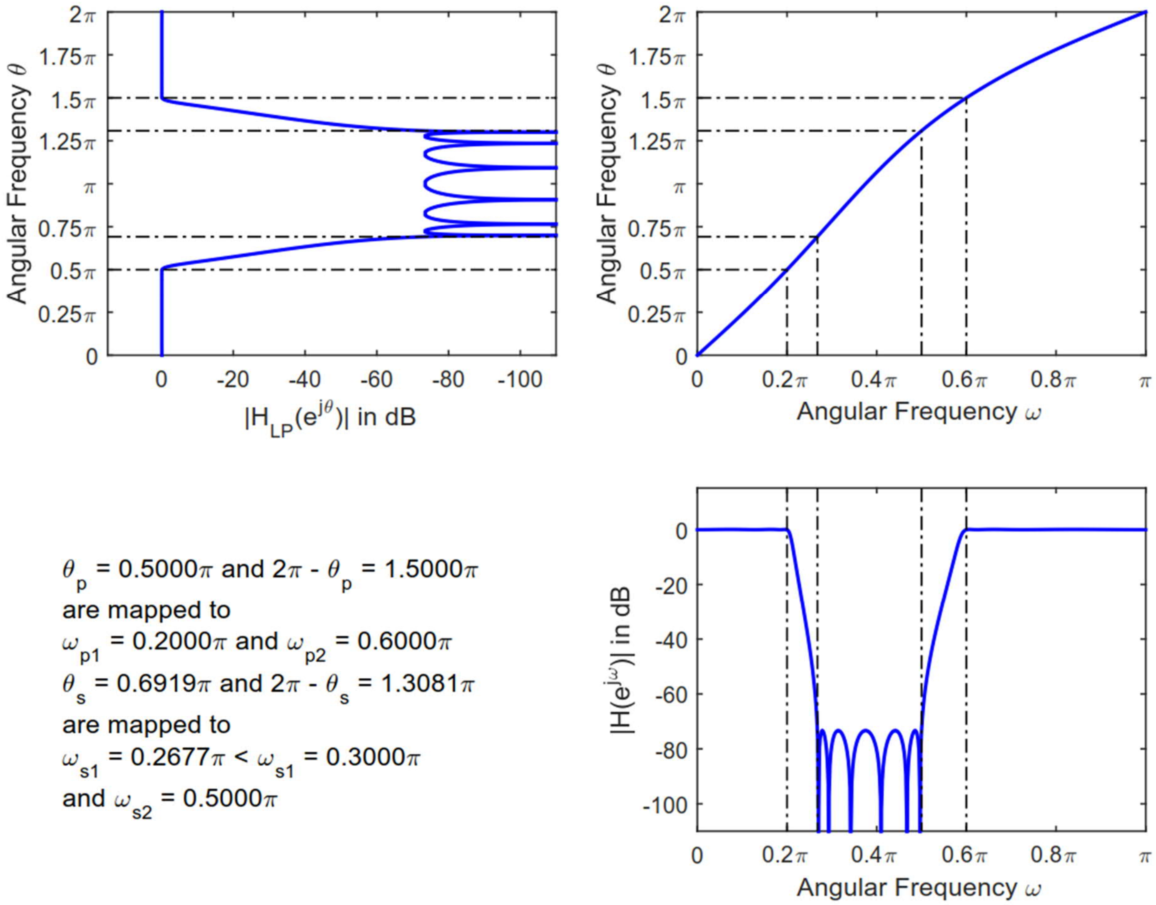

Figure 2. Design of a band-stop filter with $\Omega_p = [0, 0.2\pi] \cup [0.6\pi, \pi]$ and $\Omega_s = [0.3\pi, 0.5\pi]$ using a low-pass to band-pass transformation for $\theta_p = \pi/2$. $\theta_s = 0.691863\pi$ is required.

The key difference compared with the band-pass case is that the region $[0, 2\pi]$ of $|H_{\mathrm{LP}}(e^{j\theta})|$, instead of $[-\pi, \pi]$, is converted onto the region $[0, \pi]$ of $|H(e^{j\omega})|$. Moreover, $\theta_p = -\pi/2$ and $2\pi - \theta_p = 3\pi/2$ are, respectively, mapped to $\omega_{p1} = 0.2\pi$ and $\omega_{p2} = 0.6\pi$, and, in terms of the function $f_{BS}(\omega)$ in (33b), the condition for θ_s is

$$\theta_s = \min\{f_{BS}(\omega_{s1}), 2\pi - f_{BS}(\omega_{s2})\}. \tag{36}$$

In the case of Fig. 2, $\theta_s = 0.691863\pi$ and $2\pi - \theta_s = 1.308134$ is mapped to $\omega_{s2} = 0.5\pi$. θ_s, in turn, is mapped to $\omega = 0.2677205\pi < \omega_{s1} = 0.3\pi$ so that the first stopband edge angle smaller than needed.

For designing high-pass filters, the transformation

$$Z^{-1} = -z^{-1}. \tag{37}$$

is included in Table III. In this case, the overall synthesis is extremely simple, as will be seen in the following.

III. Simplified Design of High-Pass, Band-Pass and Band-Stop Filters

Given the criteria for the transfer function

$$H(z) = \kappa \prod_{\ell=1}^{N} (1 - \alpha_\ell z^{-1}) \Big/ \prod_{\ell=1}^{N} (1 - \beta_\ell z^{-1}). \tag{38}$$

in the form

$$\begin{aligned} &0 \leq \text{Loss}\left(H\left(e^{j\omega}\right)\right) \leq A_p \ \text{ for } \omega \in \Omega_p \\ &A_s \leq \text{Loss}\left(H\left(e^{j\omega}\right)\right) \ \text{ for } \omega \in \Omega_s, \end{aligned} \tag{39}$$

where the passband and stopband regions Ω_p and Ω_s are specified according to Table I for the high-pass, band-pass, and band-stop filters, the filter synthesis can be carried out in the following three steps:

Step I: Determine the Prototype Filter Criteria

In the high-pass case, select

$$\theta_p = \pi - \omega_p \text{ and } \theta_s = \pi - \omega_s. \tag{40}$$

In the band-pass and band-stop cases, select

$$\theta_p = \pi/2. \tag{41}$$

Moreover, determine in the band-pass and band-stop cases, respectively,

$$\theta_s = \min\{|f_{BP}(\omega_{s1})|, f_{BP}(\omega_{s2})\}, \tag{42}$$

and

$$\theta_s = \min\{f_{BS}(\omega_{s1}), 2\pi - f_{BS}(\omega_{s2})\}, \tag{43}$$

where $f_{BP}(\omega)$ and $f_{BS}(\omega)$ are defined by (33a), (33b), (34a), and (34b).

Step II: Synthesis of the Prototype Filter

Design the prototype filter with the transfer function of the form

$$H_{\text{LP}}(Z) = \mathrm{K} \prod_{\ell=1}^{M} (1 - \mathrm{A}_\ell Z^{-1}) \Big/ \prod_{\ell=1}^{M} (1 - \mathrm{B}_\ell Z^{-1}) \tag{44}$$

to meet the specifications

$$0 \leq \text{Loss}\left(H_{\text{LP}}\left(e^{j\theta}\right)\right) \leq A_p \quad \text{for } \theta \in \left[0, \theta_p\right]$$
$$A_s \leq \text{Loss}\left(H_{\text{LP}}\left(e^{j\theta}\right)\right) \quad \text{for } \theta \in [\theta_s, \pi]. \tag{44}$$

This can be performed using the techniques introduced in Part I of this contribution.

Step III: Conversion of the Prototype Filter to the Desired Filter

In the high-pass case, where the transformation $Z^{-1} = -z^{-1}$ is in use, $\kappa \equiv \mathrm{K}$, $N \equiv M$, and

$$\alpha_\ell = -\mathrm{A}_\ell \text{ and } \beta_\ell = -\mathrm{B}_\ell \text{ for } \ell = 1, 2, \ldots, N. \tag{45}$$

In the band-pass and band-stop cases, the transformations are, respectively,

$$Z^{-1} = -\frac{z^{-2} - \gamma_1 z^{-1} + \gamma_2}{1 - \gamma_1 z^{-1} + \gamma_2 z^{-2}}$$

and

$$Z^{-1} = \frac{z^{-2} - \gamma_1 z^{-1} + \gamma_2}{1 - \gamma_1 z^{-1} + \gamma_2 z^{-2}}.$$

Hence, corresponding to each of the poles at $Z = \mathrm{B}_\ell$ for $\ell = 1, 2, \ldots, M$, there are two poles that are roots of the equations

$$(\gamma_2 \mathrm{B}_\ell + 1) z^2 - \gamma_1 (\mathrm{B}_\ell + 1) + (\mathrm{B}_\ell + \gamma_2) = 0 \tag{46}$$

and

$$(\gamma_2 \mathrm{B}_\ell - 1) z^2 - \gamma_1 (\mathrm{B}_\ell - 1) + (\mathrm{B}_\ell - \gamma_2) = 0, \tag{47}$$

respectively.

Similarly, for each zero at $Z = \mathrm{B}_\ell$, for $\ell = 1, 2, \ldots, M$, there are two zeros that are roots of the equations, which are obtained from the above two equations by replacing B_ℓ byA_ℓ. Since the number of poles and zeros is doubled, the order of the resulting filter is twice that of the prototype, that is, $N = 2M$. Finally, it can be shown, after some reasoning, that

$$\kappa = \mathrm{K} \prod_{\ell=1}^{M} (1 + \mu \mathrm{A}_\ell) \bigg/ \prod_{\ell=1}^{M} (1 + \mu \mathrm{B}_\ell), \tag{48}$$

where $\mu = \gamma_2$ and $\mu = -\gamma_2$for the band-pass and band-stop cases, respectively, preserves the magnitude values in the overall transformation..